Math 409 Midterm 2

Name: ____________________

This exam has 6 questions, for a total of 100 points + 16 bonus points.

Please answer each question in the space provided. No aids are permitted.

Question 1. (40 pts)

In each of the following eight cases, indicate whether the given statement is true or false. No justification is necessary.

(a) Every subsequence of a Cauchy sequence is also a Cauchy sequence.

Solution: True.

(b) If f is a continuous function on a closed and bounded interval I, then there exists $x_0 \in I$ such that $f(x_0)$ is the maximum of f on I.

Solution: True.

(c) $\lim\limits_{x \to 0} \dfrac{x^3 \sin(1/x) + x}{x \cos x}$ exists.

Solution: True.

(d) A bounded sequence $\{x_n\}$ in $\mathbb{R}$ can have two subsequences converging to two different numbers.

Solution: True.

(e) If $g(x) \leq -1$ for all $x \in \mathbb{R}$ and $\lim_{x \to a} f(x) = 0$, then $\displaystyle\lim_{x \to a} \frac{g(x)}{f(x)} = -\infty$.

Solution: False.

(f) A bounded increasing sequence converges to a finite number.

Solution: True.

(g) If $f : \mathbb{R} \to \mathbb{R}$ is a uniformly continuous function, then f is bounded on $\mathbb{R}$.

Solution: False.

(h) Given a sequence $\{x_n\}$ with $x_n > 0$ for all n, if $\{x_n\}$ has no converging subsequences, then $x_n \to \infty$, as $n \to \infty$.

Solution: True.

Question 2. (20 pts)

(a) State the definition of Cauchy sequences.

Solution: Omitted. You can find it in the textbook.

(b) Let $\{x_n\}$ be a real sequence such that

$$x_{n+1} = x_n + \left(\frac{1}{3}\right)^n.$$

Prove that $\{x_n\}$ is a Cauchy sequence.

Solution: For any $n > m$, we have

$$\begin{aligned}
|x_n - x_m| &= |x_n - x_{n-1} + x_{n-1} - x_{n-2} + \cdots - x_{m+1} + x_{m+1} - x_m| \\
&\leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m| \\
&= \left(\frac{1}{3}\right)^{n-1} + \left(\frac{1}{3}\right)^{n-2} + \cdots + \left(\frac{1}{3}\right)^{m} \\
&= \frac{(1/3)^m(1-(1/3)^{n-m})}{1-(1/3)} \leq (3/2)\cdot(1/3)^m
\end{aligned}$$

So for any $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that $(3/2)\cdot(1/3)^N < \varepsilon$. Then for any $n, m > N$, we have

$$|x_n - x_m| < (3/2)\cdot(1/3)^N < \varepsilon.$$

So $\{x_n\}$ is Cauchy.

Question 3. (20 pts)

(a) State the Intermediate Value Theorem.

> **Solution:** Omitted. You can find it in the textbook.

(b) Prove that there exists an $x \in \mathbb{R}$ such that $4^x = x^3 + \sin x + x^2 + 2$.

> **Solution:** Consider the function $g(x) = 4^x - (x^3 + \sin x + x^2 + 2)$. On the interval $[0, 3]$, we have
>
> $$g(0) = -1 \text{ and } g(3) = 64 - (27 + \sin 3 + 9 + 2) > 0.$$
>
> So we have $g(0) \leq 0 \leq g(3)$. It follows from the intermediate value theorem that there exists $x_0 \in [0, 3]$ such that $g(x_0) = 0$.

Question 4. (20 pts)

(a) State the definition of uniform continuity.

Solution: Omitted. You can find it in the textbook.

(b) Let f and g be uniformly continuous functions on $\mathbb{R}$. Prove that $f+g$ is uniformly continuous on $\mathbb{R}$.

Solution: Since f is uniformly continuous on $\mathbb{R}$, for any $\varepsilon > 0$, there exists δ_1 such that

$$|f(x) - f(y)| < \varepsilon/2$$

for all $x, y \in \mathbb{R}$ with $|x - y| < \delta_1$. Similarly, there exists δ_2 such that

$$|g(x) - g(y)| < \varepsilon/2$$

for all $x, y \in \mathbb{R}$ with $|x - y| < \delta_2$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then we have

$$|(f+g)(x) - (f+g)(y)| \leq |f(x) - f(y) + |g(x) - g(y)| < \varepsilon$$

for all $x, y \in \mathbb{R}$ with $|x - y| < \delta$.

(c) Let f and g be uniformly continuous functions on $\mathbb{R}$. If both f and g are bounded on $\mathbb{R}$, then fg is also uniformly continuous on $\mathbb{R}$.

Solution: Since f and g are bounded over $\mathbb{R}$, there exists $M > 0$ such that

$$|f(x)| < M \text{ and } |g(x)| < M$$

for all $x \in \mathbb{R}$.
Since f is uniformly continuous on $\mathbb{R}$, for any $\varepsilon > 0$, there exists δ_1 such that

$$|f(x) - f(y)| < \varepsilon/2M$$

for all $x, y \in \mathbb{R}$ with $|x - y| < \delta_1$. Similarly, there exists δ_2 such that

$$|g(x) - g(y)| < \varepsilon/2M$$

for all $x, y \in \mathbb{R}$ with $|x - y| < \delta_2$.
Let $\delta = \min\{\delta_1, \delta_2\}$. Then for all $x, y \in \mathbb{R}$ with $|x - y| < \delta$, we have

$$\begin{aligned}|f(x)g(x) - f(y)g(y)| &= |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)| \\ &\leq |f(x)g(x) - f(x)g(y)| + |f(x)g(y) - f(y)g(y)| \\ &= |f(x)||g(x) - g(y)| + |g(y)||f(x) - f(y)| < \varepsilon.\end{aligned}$$

Bonus Question 5. (8 pts)

If $\{x_n\}$ is a Cauchy sequence, prove that $\{x_n^2\}$ is also a Cauchy sequence.

> **Solution:** If $\{x_n\}$ is a Cauchy sequence, then $\{x_n\}$ converges to some finite real number L. It follows that $\{x_n^2\}$ also converges to L^2. So $\{x_n^2\}$ is Cauchy.

Bonus Question 6. (8 pts)

If f and g are uniformly continuous functions on the interval $(0, 1)$, prove that fg is uniformly continuous on $(0, 1)$.

> **Solution:** If f and g are uniformly continuous functions on $(0, 1)$, then f and g can be extended to continuous functions on $[0, 1]$.
>
> Now it follows that f and g are bounded on $(0, 1)$. Now the same argument from Question 4 part (c) can be used to prove fg is uniformly continuous on $(0, 1)$.
>
> Alternatively, since f and g are now continuous on $[0, 1]$, it follows that fg is continuous on $[0, 1]$. Hence fg is uniformly continuous on $[0, 1]$, and thus uniformly continuous on $(0, 1)$.